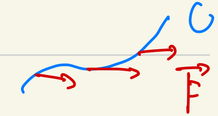
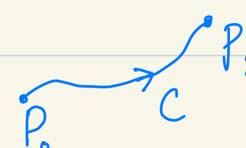
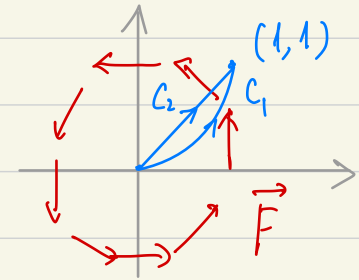
§ Line Integral

	Scalar function $\int_C f\,ds$	Vector Field $\int_C \vec{F}\cdot d\vec{r}$
Interpretation	Mass of wire C w/ density $\delta = f$	Work of $\vec{F}$ along C 
Computation via parametrization C: $\vec{r}(t)$, $a \le t \le b$	$\int_C f\,ds = \int_a^b f(\vec{r}(t)) \left\lvert \frac{d\vec{r}}{dt} \right\rvert dt$	$\int_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(t)\cdot \frac{d\vec{r}}{dt}\,dt \overset{\mathbb{R}^2:\ \vec{F} = \langle P, Q\rangle}{=} \int_C P\,dx + Q\,dy$
Fundamental Thm of Calculus for line integral	N.A.	If $\vec{F} = \nabla f$ gradient field $\boxed{\int_C \nabla f\cdot d\vec{r} = f(P_1) - f(P_0)}$



Example 1: $\vec{F} = -y\hat{i} + x\hat{j} = \langle -y, x\rangle$

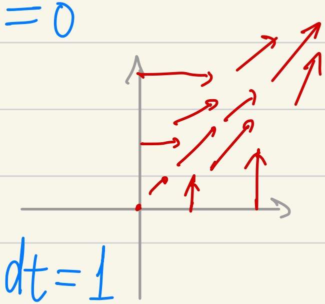


C_1: $\vec{r}(t) = \langle t, t^2\rangle$, $\qquad \vec{F} = \langle -y, x\rangle = \langle -t^2, t\rangle$

$$\Rightarrow \int \vec{F}\cdot d\vec{r} = \int_0^1 \overbrace{\langle -t^2, t\rangle}^{\vec{F}(t)} \cdot \overbrace{\langle 1, 2t\rangle}^{\vec{r}\,'(t)}\, dt = \int_0^1 t^2\, dt = \frac{1}{3}$$

C_2: $\vec{r}(t) = \langle t, t\rangle$, $\vec{F} = \langle -t, t\rangle$. $\quad \int_{C_2} \vec{F}\cdot d\vec{r} = \int_0^1 \langle -t, t\rangle\cdot\langle 1, 1\rangle\, dt = 0$

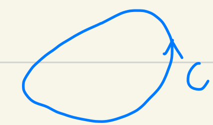
Example 2: $\vec{F} = y\hat{i} + x\hat{j} = \langle y. x\rangle = \nabla f$. $\qquad f = xy$.



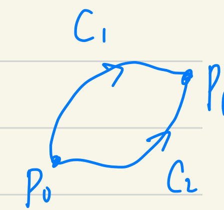
$$\int_{C_1} \vec{F}\cdot d\vec{r} = \int_0^1 \langle t^2, t\rangle \langle 1, 2t\rangle\, dt = 1. \qquad \int_{C_2} \vec{F}\cdot d\vec{r} = \int_0^1 \langle t, t\rangle\cdot\langle 1, 1\rangle\, dt = 1$$

$$f(1.1) - f(0,0) = 1.$$

Equivalent Properties: $\vec{F}$ vector field on Ω



(1) $\vec{F}$ is conservative: $\oint_C \vec{F}\cdot d\vec{r} = 0$ $\forall$ closed curve $C \subset D$



(2) $\vec{F}$ is path-independence: $\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_2} \vec{F}\cdot d\vec{r}$ $\forall C_1, C_2 \subset D$ w/ same end points

(3) $\vec{F} = \nabla f$ gradient field.

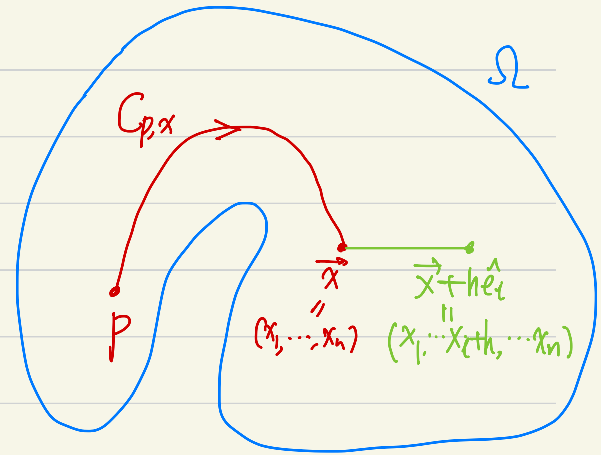
pf: (1) $\Leftrightarrow$ (2) trivial

(2) $\Leftarrow$ (3): Fund Thm of Calculus for line integrals: $\boxed{\int_C \nabla f \cdot d\vec{r} = f(P_1) - f(P_0)}$

(2) $\Rightarrow$ (3): Find potential using line integrals.

Suppose $\vec{F}$ path indep. on a connected domain $\Omega \subset \mathbb{R}^n$

Define $f(x) = \int_{C_{p,x}} \vec{F} \cdot d\vec{r}$ for any curve $C_{p,x}$ and basepoint P.



- Path indep $\Rightarrow f$ well-defined
- To prove $\nabla f = \vec{F}$:

$\vec{r}(t) = \vec{x} + t \cdot \hat{e}_i$

$$\frac{\partial f}{\partial x_i}(x) = \lim_{h\to 0} \frac{f(x+h\hat{e}_i) - f(x)}{h} = \lim_{h\to 0} \frac{1}{h} \int_{C_{\vec{x}, \vec{x}+h\hat{e}_i}} \vec{F} \cdot d\vec{r} = \lim_{h\to 0} \frac{1}{h} \int_0^h \underbrace{\vec{F}(\vec{x}+t\hat{e}_i) \cdot \hat{e}_i}_{F_i(\vec{x}+t\hat{e}_i)} \, dt$$

Fund Thm of Calculus

$$= F_i(x)$$

$\square$

also SUFFICIENT condition in many cases

Example: Find potential of $\vec{F} = \langle 4x^2 + 8xy,\ 3y^2 + 4x^2 \rangle$.

Note: If $\vec{F} = \langle P, Q \rangle = \nabla f = \langle f_x, f_y \rangle$, then NECESSARY condition $P_y = Q_x \Leftrightarrow f_{xy} = f_{yx}$

Two methods:

1) Antiderivative: Want to solve $\begin{cases} f_x = 4x^2 + 8xy & (1) \\ f_y = 3y^2 + 4x^2 & (2) \end{cases}$

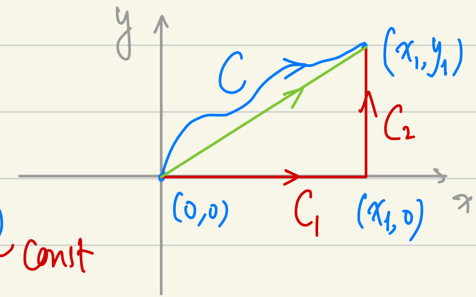
Antiderivative w.r.t. x

$(1) \Rightarrow f = \frac{4}{3}x^3 + 4x^2y + g(y)$ ← integration "constant": indep of x.

Hence $f_y = 4x^2 + g'(y)$ then $(2) \Rightarrow g'(y) = 3y^2 \Rightarrow g(y) = y^3 + C$ const.

Plug back, $f = \frac{4}{3}x^3 + 4x^2y + y^3 + C$.

2) Computing line integral.



$$\int_C \vec{F}\cdot d\vec{r} = f(x_1,y_1) - f(0,0) \Rightarrow f(x_1,y_1) = \int_C \vec{F}\cdot d\vec{r} + \underbrace{f(0,0)}_{\text{const}}$$

On C_1: $\displaystyle\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_1} P\,dx + Q\,\cancelto{0}{dy} = \int_0^{x_1} 4x^2 + \underbrace{\cancel{8xy}}_{y=0}\,dx = \frac{4}{3}x_1^3$

On C_2: $\displaystyle\int_{C_2} \vec{F}\cdot d\vec{r} = \int_{C_2} P\,\cancelto{0}{dx} + Q\,dy = \int_0^{y_1} 3y^2 + 4\underbrace{x^2}_{=x_1^2}\,dy = y^3 + 4x_1^2 y\Big|_0^{y_1} = y_1^3 + 4x_1^2 y_1$

$$\Rightarrow f(x,y) = \int_{C_1} + \int_{C_2} + f(0,0) = \frac{4}{3}x^3 + y^3 + 4x^2y + C$$

§ Green's Thm

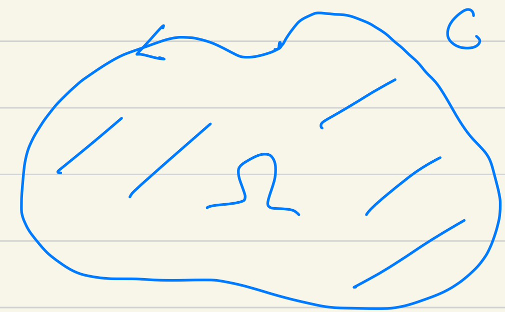
From now on, $\vec{F} = \langle P, Q \rangle$ field in 2-dim region $\Omega \subset \mathbb{R}^2$

Green's Theorem: If C is a simple closed curve enclosing a region Ω
(no self-intersection)

$\vec{F} = \langle P, Q \rangle$ vector field defined everywhere

then
$$\boxed{\oint_C P\,dx + Q\,dy = \iint_\Omega Q_x - P_y \, dA}$$

$$\boxed{\oint_C \vec{F} \cdot d\vec{r} = \iint_R \operatorname{curl} \vec{F} \, dA}$$

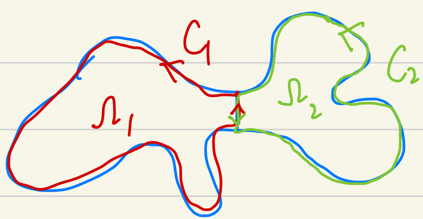


Pf of Green's thm: 1) We'll prove $\oint_C P\,dx = \iint_\Omega -P_y\,dA$ (Special Case $Q=0$)

Similarly, $\oint_C Q\,dy = \iint_\Omega Q_x\,dA$ (Special Case $P=0$)

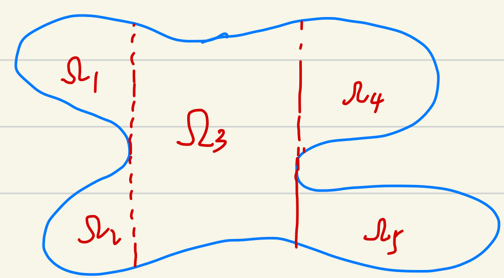
Summing, get Green's thm

2) Decompose Ω into Simpler region.



If we prove $\oint_{C_i} P\,dx = \iint_{R_i} -P_y\,dA$

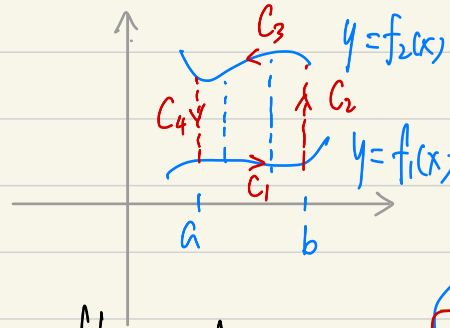
then $\oint_C = \oint_{C_1} + \oint_{C_2} = \iint_{\Omega_1} + \iint_{\Omega_2} = \iint_\Omega$



Cut Ω into "Vertically simple region": $a < x < b$
$f_1(x) < y < f_2(x)$.

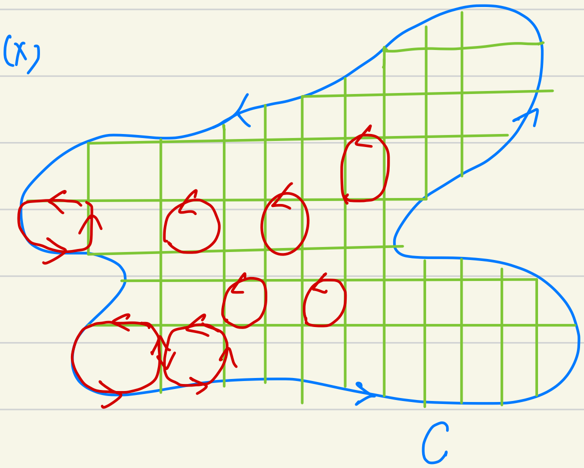
3) $\int_{C_1} P\,dx = \int_a^b P(x, y=f_1(x))\,dx$

$\int_{C_2} P\,dx = \int_{C_4} P\,dx = 0$



$\int_{C_3} P\,dx = \int_b^a P(x, y=f_2(x))\,dx = -\int_a^b P(x, f_2(x))\,dx$

$\Rightarrow \oint_C P\,dx = \int_a^b \big(P(x, f_1(x)) - P(x, f_2(x))\big)\,dx$



• RHS: $\int_\Omega -P_y\,dA = -\int_a^b \left(\int_{f_1(x)}^{f_2(x)} \frac{\partial P}{\partial y}\,dy \right) dx$

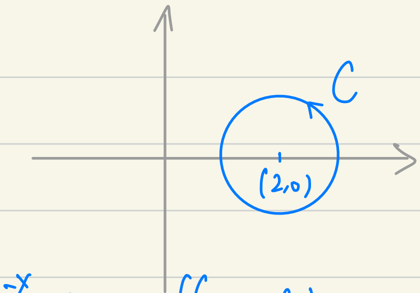
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \|$

$$P(x,y)\Big|_{y=f_1(x)}^{y=f_2(x)} = P(x, f_2(x)) - P(x, f_1(x)).$$

□

Ex 1: $\vec{F} = \langle y e^{-x}, \tfrac{1}{2}x^2 - e^{-x} \rangle$

C = circle of radius 1 centered at $(2,0)$



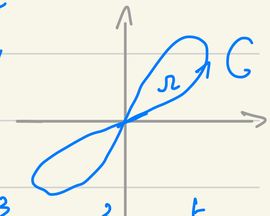
Green's thm $\Rightarrow \oint_C \vec{F}\cdot d\vec{r} = \iint_\Omega \operatorname{curl}\vec{F}\, dA = \iint_\Omega (x+e^{-x}) - e^{-x}\, dA = \iint_\Omega x\, dA = 2\pi$

- Find $\operatorname{Area}(\Omega) = \iint_\Omega \underbrace{1}_{\operatorname{curl}\vec{F}}\, dA = \oint_C \vec{F}\cdot d\vec{r}$

Possible $\vec{F}$: $\langle 0, x \rangle$, $\langle -y, 0 \rangle$, $\langle -\frac{y}{2}, \frac{x}{2} \rangle$

$$\Rightarrow \boxed{\operatorname{Area}(\Omega) = \oint_C x\,dy = \oint_C -y\,dx = \frac{1}{2}\oint -y\,dx + x\,dy}$$

Ex 2: Find area of "propeller"

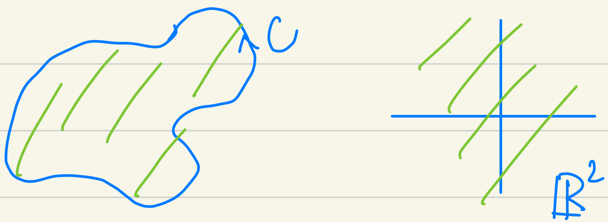


enclosed by $\vec{r}(t) = \langle t - t^3, 2t^3 - 2t^5 \rangle$ $(-1 \le t \le 1)$

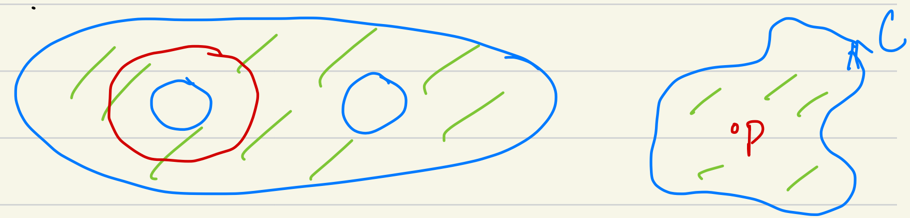
$$\text{Area} = 2\operatorname{Area}(\Omega) = 2\oint x\,dy$$
$$= 2\cdot\int_0^1 (t-t^3)\cdot(6t^2 - 10t^4)\,dt$$
$$= \frac{1}{6}$$

$\square$

Def: A region Ω is called **Simply-Connected** if the interior of any closed curve in Ω is also contained in Ω. "no holes"



Simply-Connected



NOT Simply-Connected

Thm: Suppose $\vec{F} = \langle P, Q \rangle$ smooth field on Ω, Simply Connected

TFAE:

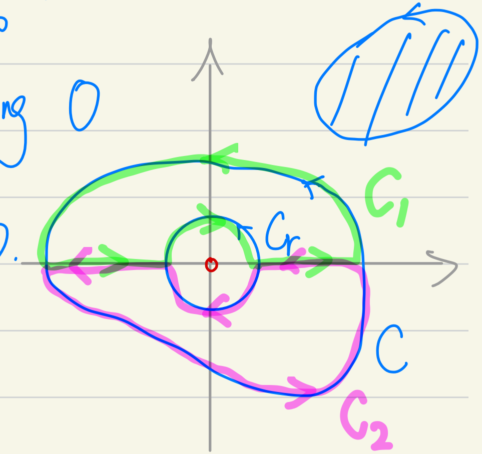
(1) $\vec{F}$ is gradient (2) $\vec{F}$ is path-indep

(3) $\vec{F}$ Conservative (4) $\text{curl}\,\vec{F} = 0 \iff Q_x - P_y = 0$

(4) $\Rightarrow$ (3): Green's Thm.

What if Ω not simply-connected? (e.g. $\vec{F}$ not defined everywhere inside C?)

Ex: $\vec{F} = \langle -\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2} \rangle$ not defined at 0: $\lim_{x,y\to 0} |\vec{F}| = \infty$.



- $\text{Curl}\,\vec{F} = 0 \Rightarrow \oint_{C'} \vec{F}\cdot d\vec{r} = 0$ for C' not enclosing 0

- However, $\oint_C \vec{F}\cdot d\vec{r}$ can't use Green directly if C encloses 0.

In fact, Suppose $C_r = \langle r\cos t, r\sin t \rangle$.

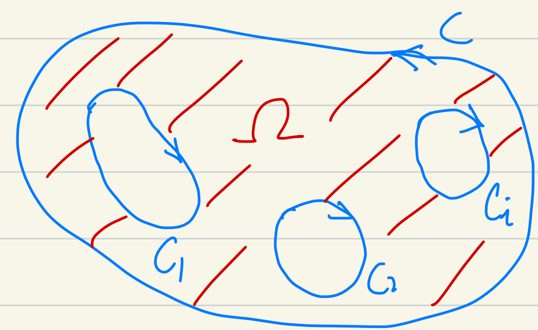
$$\oint_{C_r} \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \langle -\frac{\sin t}{r}, \frac{\cos t}{r} \rangle \cdot \langle -r\sin t, r\cos t \rangle \, dt = 2\pi \neq 0.$$

NOT CONSERVATIVE!

Note: $\oint_C \vec{F}\cdot d\vec{r} - \underbrace{\oint_{C_r} \vec{F}\cdot d\vec{r}}_{=\,2\pi} = \oint_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r} = \iint_{\Omega_1+\Omega_2} \cancelto{0}{\operatorname{curl}\vec{F}} \cdot d\vec{A} = 0$

$\Rightarrow \oint_C \vec{F}\cdot d\vec{r} = 2\pi \quad \forall\, C$ enclosing O

Green's Thm (Refined Version): Suppose Ω bounded by $C, C_1, \cdots, C_i$



$\vec{F}$ smoothly defined in Ω

Then
$$\boxed{\oint_{\underbrace{C-C_1-\cdots-C_i}_{\partial\Omega\ \text{(oriented) boundary of }\Omega}} \vec{F}\cdot d\vec{r} = \iint_\Omega \operatorname{curl}\vec{F}\cdot d\vec{A}}$$